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A Note on Closed-to-Convex Functions

(An Application of Schwarz-Christoffel Formula)

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Let $z_k : k=1,2,\dots,n$ be points on the unit circle $|z|=1$ such that

$$z_k = e^{i\theta_k} \quad (0 \leq \theta_1 < \theta_2 < \dots < \theta_n < 2\pi)$$

and $w(z)$ defined in the unit disc $|z| < 1$ be the function which satisfies the following equation

$$(1) \quad \frac{dw}{dz} = C (z-z_1)^{\alpha_1-1} (z-z_2)^{\alpha_2-1} \dots (z-z_n)^{\alpha_n-1}$$

where C is a constant complex number, and $\alpha_k (k=1,2,\dots,n)$ satisfy

$$0 \leq \alpha_k \leq 2, \quad \sum_{k=1}^n \alpha_k = n-2$$

and $(*)$ are assumed to take values of the branch $1^{2\pi} = 1$.

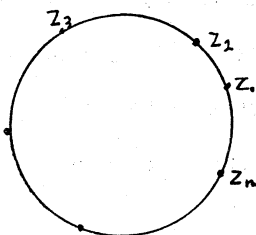


Fig. 1

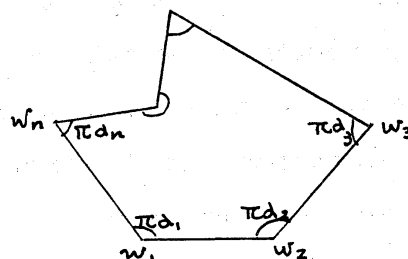


Fig. 2

If points z_k on the unit circle $|z|=1$ are suitably chosen, by the function $w(z)$, the interior of unit disc in z -plane is transformed into the interior of a polygon with n sides in the w -plane, and each vertex w_k of the polygon corresponds to z_k on the unit circle $|z|=1$. And each interior angle at w_k is equal to $\pi\alpha_k$.

But if points z_k on the unit circle are arbitrary chosen, the polygon defined by (1) may be not in general a bounded polygon in common sense. This formula (1) is well-known as Schwarz-Christoffel's transformation.

The function $w(z)$ defined in (1) can be normalised to $w(0) = 0$ and $w'(0) = 1$, as follows

$$(2) \frac{dw}{dz} = (1 - \varepsilon_1 z)^{\delta_1} (1 - \varepsilon_2 z)^{\delta_2} \cdots (1 - \varepsilon_n z)^{\delta_n}$$

where $\varepsilon_k = z_k^{-1}$, $\delta_k = \alpha_k - 1$, $|\delta_k| \leq 1$ and $\sum_{k=1}^n \delta_k = -2$.

At first, we consider the case when a polygon is convex. In this case, interior angles $\pi \alpha_k$ at all vertices are smaller than π , and all δ_k satisfy

$$-1 < \delta_k < 0, \quad \sum_{k=1}^n \delta_k = -2$$

And we have a following relation from (2)

$$z \frac{w''(z)}{w'(z)} = - \sum_{k=1}^n \frac{\delta_k \varepsilon_k z}{1 - \varepsilon_k z} = - \frac{1}{2} \sum_{k=1}^n \delta_k \frac{1 + \varepsilon_k z}{1 - \varepsilon_k z} - 1$$

and from $\delta_k < 0$, we have an equality

$$\operatorname{Re.} z \frac{w''(z)}{w'(z)} > -1 : |z| < 1$$

Accordingly, in the case when all δ_k are negative, for arbitrary points z_k on the unit circle, the function $w(z)$ defined by (2) is a convex function.

Next, we consider the estimation of coefficients by the Taylor expansion of a convex function $w(z)$ defined by (2). Generally, when γ is a positive number, in the power series

$$(1 - z)^{-\gamma} = 1 + \frac{\gamma}{1!} z + \frac{\gamma(\gamma+1)}{2!} z^2 + \frac{\gamma(\gamma+1)(\gamma+2)}{3!} z^3 + \cdots$$

all coefficients are positive.

Accordingly, all coefficients of power series of $(1 - \varepsilon_k z)^{\delta_k}$ in (2) are majorated by coefficients of power series of $(1 - z)^{\delta_k}$, and coefficients of the power series of dw/dz in (2) are majorated by coefficients of

$$\begin{aligned} (1 - z)^{\delta_1} (1 - z)^{\delta_2} \cdots (1 - z)^{\delta_n} &= (1 - z)^{-2} \\ &= 1 + 2z + 3z^2 + 4z^3 + \cdots \end{aligned}$$

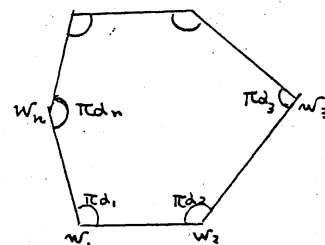


Fig. 3

and all coefficients of Taylor expansion of $w(z)$ are majorated by coefficients of

$$(1-z)^{-1} = 1 + z + z^2 + z^3 + \dots$$

Next, we consider the case when a function defined by (2) is closed-to-convex. At first, we show a lemma as follows

Lemma. Let z_k ($k=1, 2, \dots, 2n$) be points on the unit circle $|z|=1$ such that

$$z_k = e^{i\theta_k} : 0 \leq \theta_1 \leq \theta_2 \leq \dots \leq \theta_{2n} \leq 2\pi$$

and $\phi(z)$ be a function represented by

$$\phi(z) = \frac{1 - \varepsilon_2 z}{1 - \varepsilon_1 z} \cdot \frac{1 - \varepsilon_4 z}{1 - \varepsilon_3 z} \cdot \dots \cdot \frac{1 - \varepsilon_{2n} z}{1 - \varepsilon_{2n-1} z}$$

where $\varepsilon_k = z_k^{-1}$.

Then the function $\phi(z)$ takes values on a half plane which contains the unit in its interior, and is bordered by a line which passes the origin.

Proof. In Fig. 4, when $|z|=1$, we have

$$\arg \frac{z_2 - z}{z_1 - z} = \begin{cases} \frac{1}{2}(\theta_2 - \theta_1) & : z \in \widehat{z_1 z_2} \\ \frac{1}{2}(\theta_2 - \theta_1) + \pi & : z \in \widehat{z_1 z_2} \end{cases}$$

and when $|z| < 1$, we have

$$\frac{1}{2}(\theta_2 - \theta_1) < \arg \frac{z - z_2}{z - z_1} < \frac{1}{2}(\theta_2 - \theta_1) + \pi$$

Accordingly, when $|z|=1$, we have

$$\arg \frac{1 - \varepsilon_2 z}{1 - \varepsilon_1 z} = \begin{cases} -\frac{1}{2}(\theta_2 - \theta_1) & : z \in \widehat{z_1 z_2} \\ -\frac{1}{2}(\theta_2 - \theta_1) + \pi & : z \in \widehat{z_1 z_2} \end{cases}$$

When z varies on the unit circle, if z is not on any one of arcs $\widehat{z_1 z_2}, \widehat{z_3 z_4}, \dots, \widehat{z_{2n-1} z_{2n}}$

$$\arg \phi(z) = \Theta = -\frac{1}{2}(\theta_2 - \theta_1 + \theta_4 - \theta_3 + \dots + \theta_{2n} - \theta_{2n-1})$$

and if z is on any one of these arcs, $\arg \phi(z)$ is equal to $\Theta + \pi$.

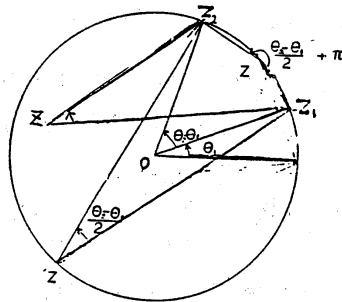


Fig. 4

And when z is an interior point on the unit disc, we have

$$\theta < \arg \phi(z) < \theta + \pi$$

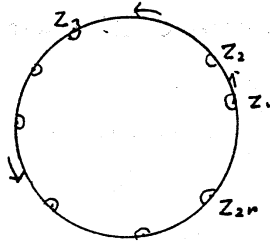


Fig. 5

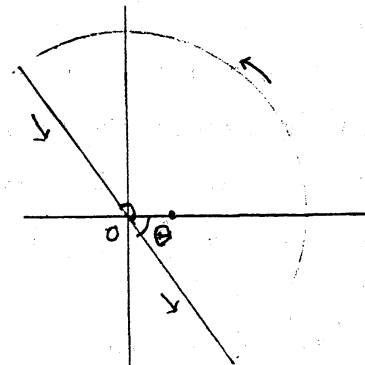


Fig. 6

Now the lemma has been proved.

Next I show a theorem as follows.

Theorem. Let $z_k = e^{i\theta_k}$: $k = 1, 2, \dots, n$ ($0 \leq \theta_1 \leq \theta_2 \leq \dots \leq \theta_n$) and $z_{j,k} = e^{i\theta_{j,k}}$: $k = 1, 2, \dots, 2n_j$ ($0 \leq \theta_{j,1} \leq \theta_{j,2} \leq \dots \leq \theta_{j,2n_j}$) ; $j=1, 2, \dots, m$ be points on the unit circle $z = 1$, and $w(z)$ defined in the unit disc $|z| < 1$ be a function which satisfies

$$(3) \quad \frac{dw}{dz} = \prod_{k=1}^n (1 - \varepsilon_k z)^{\delta_k} \prod_{j=1}^m \left[\prod_{k=1}^{n_j} \frac{1 - \varepsilon_{j,2k} z}{1 - \varepsilon_{j,2k-1} z} \right]^{\lambda_j}$$

where δ_k are negative numbers satisfying $\sum_{k=1}^n \delta_k = -2$, and λ_j are real numbers satisfying $\sum_{j=1}^m |\lambda_j| = 1$.

Then, in the Taylor expansion of the function $w(z)$, i.e.

$$w(z) = z + A_2 z^2 + \dots + A_k z^k + \dots$$

$$|A_n| \leq n \quad : \quad n = 1, 2, 3, \dots$$

and $w(z)$ is a closed-to-convex function.

Proof of Theorem. In the equation (3), $\phi(z) = \prod_{k=1}^n (1 - \varepsilon_k z)^{\delta_k}$ is the derivative of a convex function, and in the Taylor expansion of $\phi(z)$, i.e.

$$\phi(z) = 1 + a_1 z + a_2 z^2 + \dots + a_k z^k + \dots$$

all coefficient a_k : $k = 1, 2, \dots$ are majorated by coefficients of

$$(1-z)^{-2} = 1 + 2z + 3z^2 + \dots$$

And in (3), $\phi_j(z) = \prod_{k=1}^{n_j} \frac{1 - \varepsilon_{j,2k} z}{1 - \varepsilon_{j,2k-1} z}$ takes values on a half plane which

contains the unit in its interior, and is bordered by a line which passes the origin.

And $\psi_j(z) = \left[\phi_j(z) \right]^{\lambda_j} = \left[\prod_{k=1}^{n_j} \frac{1 - \varepsilon_{j,2k} z}{1 - \varepsilon_{j,2k-1} z} \right]^{\lambda_j}$ takes values on a domain,

which contains the unit in its interior, and is bordered by two lines meet at the angle $\pi \lambda_j$ in the origin.

Accordingly, the function

$$\psi(z) = \prod_{j=1}^m \psi_j(z) = \prod_{j=1}^m \left[\prod_{k=1}^{n_j} \frac{1 - \varepsilon_{j,2k} z}{1 - \varepsilon_{j,2k-1} z} \right]^{\lambda_j}$$

takes values on a half plane, which contains the unit in its interior and is bordered by a line passes the origin.

In the Taylor expansion of $\psi(z)$ i.e.

$$\psi(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \dots$$

it is well-known that all coefficient b_k are majorated by coefficients of the expansion

$$\frac{1+z}{1-z} = 1 + 2z + 2z^2 + 2z^3 + \dots$$

Accordingly, in the expansion of the function dw/dz defined in (3), i.e.

$$\frac{dw}{dz} = \phi(z) \psi(z) = 1 + a_1 z + a_2 z^2 + \dots$$

every coefficient a_k is majorated by the function

$$\frac{1+z}{(1-z)^3} = 1 + 2^2 z + 3^2 z^2 + \dots + n^2 z^{n-1} + \dots$$

and in the Taylor expansion

$$w(z) = z + A_2 z^2 + \dots + A_k z^k + \dots$$

all coefficients A_k are majorated by coefficients of the function

$$\frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \dots + n z^n + \dots$$

Thus, $|A_k| \leq n$ has been proved.

Moreover, in $dw/dz = \phi(z) \psi(z)$, $\phi(z)$ is the derivative of a convex function, and $\psi(z)$ takes values on a half plane which contains the unit in its interior, and is bordered by a line which passes the origin. Accordingly, the function $w(z)$ is a closed-to-convex function. Thus the theorem has been proved.

At last, I show a simple example of closed-to-convex functions. Let us consider a simply connected polygon Ω which has $2n$ sides parallel to the real axis or imaginary axis in the w -plane. If we call its vertices w_1, w_2, \dots, w_{2n} and denote its interior angles $\pi\alpha_1, \pi\alpha_2, \dots, \pi\alpha_{2n}$ respectively, α_k take the value $1/2$ or $3/2$,

and satisfy $\sum_{k=1}^{2n} \alpha_k = 2n - 2$.

We can construct the function $w = f(z)$ which maps the interior of unit circle $|z| < 1$ onto the interior of such a polygon by

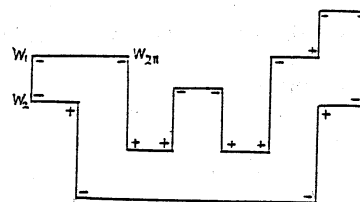


Fig. 7

$$\frac{dw}{dz} = C (z - z_1)^{\alpha_1 - 1} (z - z_2)^{\alpha_2 - 1} \dots (z - z_{2n})^{\alpha_{2n} - 1}$$

where C is a complex number, and z_k are points on the unit circle, such that

$$z_k = e^{i\theta_k}: k = 1, 2, \dots, 2n; \quad 0 < \theta_1 < \theta_2 < \dots < \theta_{2n} < 2\pi$$

Now, if we put $z_k^{-1} = \varepsilon_k$, the function $w = f(z)$ is normalized by

$$\frac{dw}{dz} = (1 - \varepsilon_1 z)^{\delta_1} (1 - \varepsilon_2 z)^{\delta_2} \dots (1 - \varepsilon_{2n} z)^{\delta_{2n}}$$

where $\delta_k (= \alpha_k - 1)$ take the value $-1/2$ or $1/2$, and $\sum_{k=1}^{2n} \delta_k = -2$.

We consider the polygon shown in Fig. 7. In this case, we can write signs of δ_k in order and if we take apart suitable four minus signs, we can arrange a sequence of couples $(-+)$ or $(+-)$ as follows

$$\ominus \ominus (+-)(-+) \ominus \ominus (-+)(+-)(-+)(+-)$$

In such a case, as the function $w(z)$ satisfies

$$\frac{dw}{dz} = \prod_{k=1}^4 (1 - \varepsilon_{1,k} z) \left[\prod_{k=1}^3 \frac{1 - \varepsilon_{2,2k} z}{1 - \varepsilon_{2,2k-1} z} \right]^2 \left[\prod_{k=1}^4 \frac{1 - \varepsilon_{3,2k} z}{1 - \varepsilon_{3,2k-1} z} \right]^{-1/2}$$

and satisfies the condition of the theorem, such a mapping function $w(z)$ is closed-to-convex, and in the Taylor expansion $w(z) = z + A_2 z^2 + A_3 z^3 + \dots$, all coefficients A_k satisfy $|A_k| \leq k$.

Remark. . Equalities $|A_k| = k$ ($k = 1, 2, \dots$)

can be satisfied only when

$$z_1 = z_2 = z_3 = z_6 = z_7 = z_8 = \varepsilon$$

$$z_4 = z_5 = -\varepsilon \quad (|\varepsilon| = 1)$$

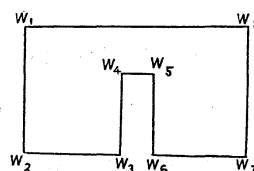


Fig. 8

as the limit case of such a polygon in Fig. 8.